Journal of Engineering Mathematics, Vol. 12, No. 4, October 1978
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in The Netherlands

A double boundary-layer model of mass transport in progressive interfacial waves

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(Received October 3, 1977 and in final form January 24, 1978)

SUMMARY

A double boundary-layer model is used to facilitate calculation of the mass transport velocity due to two-dimensional gravity waves propagating along the interface between two semi-infinite homogeneous fluids. The magnitude of this velocity in the interfacial boundary layers is found to increase with distance from the wave generating region and, in the far field, dominates the Stokes drift velocity. In the case of air above water, the mass transport in the water is such that the presence of the air may be neglected for waves of length much less than a metre, but is *dominated* by the effect of the air when the wavelength greatly exceeds a metre.

1. Introduction

The double boundary-layer theory of Stuart [1, 2] and Riley [3, 4] has recently been employed in the work of Dore [5], hereafter referred to as I, in the calculation of the mass transport velocity due to two-dimensional gravity waves in a homogeneous, viscous fluid. Thus, in I, a simplified double boundary-layer model is postulated to facilitate the calculation for Stokesian surface waves of progressive type. In the present work, this model is used to investigate the mass transport velocity due to two-dimensional waves propagating on the interface between two homogeneous fluids of great depth. In reality, the waves attenuate due to viscosity. Although such effects are treated in [5], they will be ignored in the present work, but can easily be included on a qualitative basis.

The motion is first referred to a stationary system of Cartesian co-ordinates (x, z) having origin in the mean interfacial level and z-axis directed vertically upwards. The velocity vector $\boldsymbol{q} = (u, w)$ is expanded as

$$\boldsymbol{q} = \alpha \boldsymbol{q}_1 + \alpha^2 \boldsymbol{q}_2 + \ldots, \qquad (\alpha \ll 1),$$

in terms of a small parameter α, which is the maximum wave slope in the linear inviscid theory of Stokes [6]. The second-order mass transport velocity $\alpha^2 \boldsymbol{Q}_l$ is given by Longuet-Higgins [7] as

$$\boldsymbol{Q}_l = \langle \boldsymbol{q}_2 \rangle + \left\langle \left(\left(\int \boldsymbol{q}_1 dt \cdot \nabla \right) \boldsymbol{q}_1 \right) \right\rangle = \langle \boldsymbol{q}_2 \rangle + \boldsymbol{Q}_s,$$

where $\alpha^2 \boldsymbol{Q}_s$ is the Stokes drift velocity, and $\langle\ \rangle$ denotes an average over a complete wave period. In inviscid fluids, the mass transport velocity in a progressive irrotational wave is given by

$$U_l^{(1)} = U_s^{(1)} = c\,e^{2kz}, \quad W_l^{(1)} = 0,$$

$$U_l^{(2)} = U_s^{(2)} = c\,e^{-2kz}, \quad W_l^{(2)} = 0,$$

where k is the wave-number, c is the phase velocity, and superscripts (1) and (2) denote quantities associated with the lower and upper fluids, respectively.

In slightly viscous fluid, following initiation of the progressive-wave motion, oscillatory boundary layers are quickly formed adjacent to the interface. Primary oscillatory vorticity is confined to these layers, but secondary vorticity diffuses outwards from the interface. It is shown in Section 3 that a steady-state, secondary vorticity field is established within outer boundary layers (adjacent to the oscillatory layers) which grow in the direction of wave propagation.

Of especial interest is the case of an air-water interface, for which the mass transport velocity is calculated in Section 4. It is found that, for waves whose length is a small fraction of a metre, the effect of the air is negligible, and the air-water system can be viewed as a vacuum-water system to good approximation. This confirms the results of I, where the presence of the air is neglected. For waves whose length is much greater than a metre, the influence of the air is so large that its effect dominates in the solution for the mass transport velocity.

2. Some existing results for $\rho^{(2)}/\rho^{(1)} = O(1)$, $\mu^{(2)}/\mu^{(1)} = O(1)$

We consider the case of two-dimensional motion associated with a progressive wave on the interface between two homogeneous, immiscible, incompressible fluids. The density $\rho^{(1)}$ of the lower fluid is assumed to be greater than that, $\rho^{(2)}$, of the upper fluid. It is further assumed in this section that

$$\rho^{(2)}/\rho^{(1)} = O(1), \quad \mu^{(2)}/\mu^{(1)} = O(1), \tag{2.1}$$

where μ denotes the viscosity of a fluid. Both fluids will be assumed to have infinite depth, and the progressive wave to be considered has period $2\pi/\sigma$ and wavelength $\lambda = 2\pi/k$. A stream function ψ is defined such that the velocity vector $\boldsymbol{q} = (u, w) = (\partial\psi/\partial z, -\partial\psi/\partial x)$, and the variables are non-dimensionalized according to the scheme

$$\hat{\boldsymbol{r}} = k\boldsymbol{r}, \quad \hat{t} = \sigma t, \quad \hat{\psi} = k^2\psi/\sigma, \quad \hat{\varepsilon} = (\nu k^2/\sigma)^{\frac{1}{2}},$$

where ν denotes kinematic viscosity. Then, omitting the carets, $\omega = \nabla^2\psi$ represents the fluid vorticity and the interfacial displacement is

$$z_i = \alpha e^{i(t-x)} + O(\alpha^2), \qquad (\alpha \ll 1). \tag{2.2}$$

Boundary-layer theory is assumed to be applicable in each fluid, so that we necessarily require that

$$\varepsilon^{(r)} \ll 1, \qquad (r = 1, 2). \tag{2.3}$$

Investigation of the interfacial boundary layers is facilitated by use of the orthogonal curvilinear co-ordinate system described by Longuet-Higgins [7]. Thus, s denotes arc-length measured along the interface, and n is measured positive along a normal into the *upper* fluid. The quantities s, n are used below, following their non-dimensionalization according to the above scheme.

Within the oscillatory boundary layers of thickness $O(\varepsilon)$ adjacent to the interface (assumed to be uncontaminated), we write

$$N = n/2^{\frac{1}{2}}\varepsilon, \quad \psi = \alpha\psi_{(1)}(s, N, t) + \alpha^2\psi_{(2)}(s, N, t) + \ldots \tag{2.4}$$

Regarding the mean motion within the layers, it is readily shown from the work of Dore [8] for an arbitrary oscillatory motion that

$$\left[\mu \frac{\partial^2 \Psi_l}{\partial n^2}\right] = \mu^{(1)} \frac{\partial^2 \Psi_l^{(1)}}{\partial n^2}\bigg|_{n_\infty^{(1)}} - \mu^{(2)} \frac{\partial^2 \Psi_l^{(2)}}{\partial n^2}\bigg|_{n_\infty^{(2)}}$$
$$\simeq \frac{1+i}{2^{\frac{1}{2}} i} \frac{\nu^{(1)\frac{1}{2}}}{\varepsilon^{(1)}} \frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}} \Delta \frac{\partial \Delta^*}{\partial s}, \tag{2.5}$$

$$\left[\frac{\partial \Psi_l}{\partial n}\right] \simeq \frac{2}{i} \frac{\Delta}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}} \left[(\rho^{(2)}\mu^{(2)})^{\frac{1}{2}} \frac{\partial q_{s(1)\infty}^{(1)*}}{\partial s}\right.$$
$$\left. + (\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} \frac{\partial q_{s(1)\infty}^{(2)*}}{\partial s}\right] + \frac{3}{4} \frac{1+i}{i}$$
$$\times \frac{\rho^{(1)}\mu^{(1)} - \rho^{(2)}\mu^{(2)}}{[(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}]^2} \Delta \frac{\partial \Delta^*}{\partial s}, \tag{2.6}$$

where $\Delta = q_{s(1)\infty}^{(1)} - q_{s(1)\infty}^{(2)} = [q_{s(1)}]$ denotes the change in the tangential component of velocity across the whole region of the oscillatory interfacial boundary layers, and Ψ_l is a stream function for the mass transport velocity (cf. [7]). The asterisk denotes a complex conjugate, and

$$q_{s(1)\infty}^{(1)} = e^{i(t-s)}, \quad q_{s(1)\infty}^{(2)} = -e^{i(t-s)}.$$

The results (2.5), (2.6) form boundary conditions on the mean motion outside the oscillatory layers. For finite depths of each fluid, Dore [8, 9] has applied these results to progressive waves of such small amplitude that the mean motion outside the oscillatory layers (where $\psi = \alpha\Psi_1 + \alpha^2\Psi_2 + \ldots$) is governed by the bi-harmonic equation for $\langle \Psi_2^{(r)} \rangle$. In this case, mean vorticity $O(\alpha^2\varepsilon^{-1})$ diffuses into the interior of both fluids.

In the next section, we investigate the mean motion, outside the oscillatory layers, for progressive interfacial waves of suitably large amplitude, and assume that the double boundary-layer theory of Stuart [1, 2] and Riley [3, 4] is valid. An application of this theory to progressive surface waves in deep water has been carried out in I.

3. A double boundary-layer formulation for progressive waves

It is seen from equation (2.5) that mean vorticity $O(\alpha^2\varepsilon^{-1})$ does not decay to zero at the edges of the oscillatory layers. We therefore consider the possibility that such vorticity, together with the associated mean tangential velocity from which it is derived, may decay to zero through outer interfacial boundary layers. However, the most important simplifying feature in the present calculation concerns the region of wave generation. As in the theoretical model of I, we propose to regard this region as being shrunk to a point (the origin), and accept resulting inaccuracies in the true region of wave generation.

Assuming, then, that a steady, second-order vorticity field exists outside the oscillatory, interfacial boundary layers, and using a bar to denote averages with respect to a wave period, the equation for the mean vorticity $\bar{\omega}_{(2)}$ is given by Dore [10] as

$$Q_s \frac{\partial \bar{\omega}_{(2)}}{\partial s} + Q_n \frac{\partial \bar{\omega}_{(2)}}{\partial n} = \frac{\varepsilon^2}{\alpha^2} \frac{\partial^2 \bar{\omega}_{(2)}}{\partial n^2} \tag{3.1}$$

within the assumed double boundary layers. The quantities Q_s, Q_n are the components of mass transport velocity in the orthogonal curvilinear co-ordinate system (s, n) described by Longuet-Higgins [7], and are given by

$$Q_s = \overline{\dot{s}_{(2)}} + \overline{\int \dot{s}_{(1)} dt \frac{\partial \dot{s}_{(1)}}{\partial s}} + \overline{\int \dot{n}_{(1)} dt \frac{\partial \dot{s}_{(1)}}{\partial n}},$$

$$Q_n = \overline{\dot{n}_{(2)}} + \overline{\int \dot{s}_{(1)} dt \frac{\partial \dot{n}_{(1)}}{\partial s}} + \overline{\int \dot{n}_{(1)} dt \frac{\partial \dot{n}_{(1)}}{\partial n}},$$

where $\dot{s}$, $\dot{n}$ represent the rates at which the co-ordinates of a particular fluid element are increasing. As in I, (Q_s, Q_n) is regarded as comprising an inviscid part, given by $(e^{-2n}, 0)$ and $(e^{2n}, 0)$ in the upper and lower fluid, respectively, and a viscous part.

Now the oscillatory boundary layers are established within a few oscillations of the start of the wave motion. By a consideration of the transient problem for secondary vorticity (see, also, I), the outer boundary-layer thickness increases with distance s from the origin. If we write

$$\bar{q}_s = \overline{q_s^{(i)}} + \overline{q_s^{(v)}}, \quad \bar{q}_n = \overline{q_n^{(i)}} + \overline{q_n^{(v)}}, \tag{3.2}$$

where superscript (i) signifies that a quantity corresponds to irrotational wave motion in inviscid fluid, whilst superscript (v) denotes quantities which arise due to viscosity, the thickening outer boundary layer means that $\overline{q_s^{(v)}}$ increases with s (n fixed).

3.1. *The region* $0 \leq s \leq O(\alpha^2)$

As in I, the theoretical model cannot be expected to be adequate for very small values of s. In fact, if the thickness of the outer boundary layer is $\delta(s)$, the theory breaks down in the region $0 < \delta(s) \ll \varepsilon$, where $\overline{q_{s(2)}^{(v)}} \ll 1$ and the boundary condition arising from (2.6) cannot be satisfied.

In the region where $\delta(s) = O(\varepsilon)$, $\overline{q_{s(2)}^{(v)}} = O(1)$ and the inviscid part of Q_s is approximately one within the outer boundary layer. The relevant equation in each fluid therefore has the form

$$\left(1 + \frac{\partial \overline{\psi_{(2)}^{(v)}}}{\partial n}\right) \frac{\partial^3 \overline{\psi_{(2)}^{(v)}}}{\partial s \partial n^2} - \frac{\partial \overline{\psi_{(2)}^{(v)}}}{\partial s} \frac{\partial^3 \overline{\psi_{(2)}^{(v)}}}{\partial n^3} = \frac{\varepsilon^2}{\alpha^2} \frac{\partial^4 \overline{\psi_{(2)}^{(v)}}}{\partial n^4}, \tag{3.1.1}$$

arising from equation (3.1). It is seen from (3.1.1) that $s = O(\alpha^2)$ in this region. Although the boundary value problem for $\overline{\psi_{(2)}^{(v)}}$ can be formulated, analytical solution presents considerable difficulties since, in general, no similarity solution is apparently possible on account of the non-homogeneous nature of the two coupled boundary conditions associated with (2.5) and (2.6). (An exception occurs when $\rho^{(1)}\mu^{(1)} = \rho^{(2)}\mu^{(2)}$, for then (2.6) becomes homogeneous. A similarity solution, valid for $s = O(\alpha^2)$, is readily obtained.) The boundary-layer assumptions for the region $s = O(\alpha^2)$ require that $\varepsilon \ll \alpha^2$, but no analytical or numerical investigation is attempted here.

3.2. *The region* $\alpha^2 \ll s$

Where s exceeds $O(\alpha^2)$, it is a consequence of the thickening outer boundary layer that $\overline{q_{s(2)}^{(v)}}$ is expected to become much greater than the inviscid part, $O(1)$, of Q_s. Therefore the relevant equation in each fluid is of the form

$$\frac{\partial \overline{\psi_{(2)}^{(v)}}}{\partial n} \frac{\partial^3 \overline{\psi_{(2)}^{(v)}}}{\partial s \partial n^2} - \frac{\partial \overline{\psi_{(2)}^{(v)}}}{\partial s} \frac{\partial^3 \overline{\psi_{(2)}^{(v)}}}{\partial n^3} = \frac{\varepsilon^2}{\alpha^2} \frac{\partial^4 \overline{\psi_{(2)}^{(v)}}}{\partial n^4}, \tag{3.2.1}$$

as deduced from equation (3.1). The boundary conditions are determined with the aid of (2.5) and (2.6), the kinematical condition at the interface and principles of asymptotic matching:

$$\begin{aligned}
&\mu^{(1)} \frac{\partial^2 \overline{\psi_{(2)}^{(v)(1)}}}{\partial n^2} - \mu^{(2)} \frac{\partial^2 \overline{\psi_{(2)}^{(v)(2)}}}{\partial n^2} = 2^{\frac{1}{2}} \frac{\nu^{(1)\frac{1}{2}}}{\varepsilon^{(1)}} \frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}}, && n = 0,\\
&\frac{\partial \overline{\psi_{(2)}^{(v)(1)}}}{\partial n} - \frac{\partial \overline{\psi_{(2)}^{(v)(2)}}}{\partial n} = 0, && n = 0,\\
&\overline{\psi_{(2)}^{(v)(1)}} = 0 = \overline{\psi_{(2)}^{(v)(2)}}, && n = 0, \qquad (3.2.2)\\
&\frac{\partial \overline{\psi_{(2)}^{(v)(1)}}}{\partial n} \to 0, && n \to -\infty,\\
&\frac{\partial \overline{\psi_{(2)}^{(v)(2)}}}{\partial n} \to 0, && n \to \infty.
\end{aligned}$$

It is seen from (3.2.2) that there is a mean tangential stress $O(\alpha^2\varepsilon)$, in the direction of wave propagation, acting on the fluid within the outer boundary layers. In addition to (3.2.2),

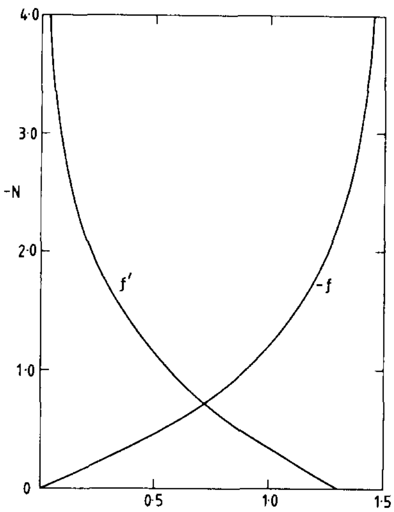

Figure 1. The solution of the boundary value problem (3.2.4).

there is an assumed known distribution of $\overline{\psi^{(v)}_{(2)}}$ at some initial section s = constant. It is evident that the boundary condition derived from (2.6) is homogeneous, and this enables us to obtain a similarity solution. Equations of the form (3.2.1) are integrated from $n = -\infty$ and $n = \infty$ for the lower and upper fluids, respectively. It is then readily shown that the resulting equations and boundary conditions admit the similarity solution

$$\overline{\psi^{(v)}_{(2)}}^{(1)} = \frac{\varepsilon^{(1)}}{\alpha^{\frac{1}{4}}}\gamma s^{\frac{3}{4}} \; f\left(\frac{\alpha^{\frac{3}{4}}}{\varepsilon^{(1)}}\frac{\gamma n}{s^{\frac{1}{4}}}\right),$$

$$\overline{\psi^{(v)}_{(2)}}^{(2)} = -\frac{\varepsilon^{(2)}}{\alpha^{\frac{1}{4}}}\gamma s^{\frac{3}{4}} \; f\left(-\frac{\alpha^{\frac{3}{4}}}{\varepsilon^{(2)}}\frac{\gamma n}{s^{\frac{1}{4}}}\right) \tag{3.2.3}$$

where

$$\gamma = \left\{\frac{2^{\frac{1}{2}}(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{[(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}]^2}\right\}^{\frac{1}{3}} = O(1),$$

and $f(N)$ satisfies the boundary value problem

$$3f''' + 2ff'' - f'^2 = 0, \quad f(0) = 0, \quad f''(0) = 1, \quad f'(-\infty) = 0, \tag{3.2.4}$$

in which $f' \equiv df/dN$, etc. A similar boundary value problem has previously arisen in the case of surface waves considered in I, where an approximate solution for an analogous function f is given. The computed solution of the boundary value problem (3.2.4) is illustrated in Fig. 1, in which $f'(0) = 1.296$ and $f(-\infty) = -1.483$.

This solution is such that the tangential component of the mass transport velocity,

$$\alpha^2 Q_s \simeq \alpha^2 \frac{\partial \overline{\psi^{(v)}_{(2)}}}{\partial n} = \alpha^{\frac{3}{2}}\gamma^2 s^{\frac{1}{2}} f', \tag{3.2.5}$$

is positive throughout the double boundary layers, and hence *is in the direction of wave propagation.* Thus, the viscous-induced contribution to the mass transport velocity quickly dominates, and is $O(\alpha^{\frac{4}{3}})$ a few wavelengths distant from the origin. Equation (3.2.5) shows that the present theory breaks down where $s = O(\alpha^{-1})$, in which region mean and primary oscillatory velocities would be comparable. Thus, validity of the similarity solution is restricted to a region where

$$\alpha^2 \ll s \ll \alpha^{-1}. \tag{3.2.6}$$

In addition, validity of the boundary-layer assumptions requires that

$$s^{\frac{2}{3}} \gg \varepsilon \alpha^{-\frac{2}{3}}. \tag{3.2.7}$$

In order that the theory should be valid where $s = O(1)$, we therefore require wave amplitudes to be sufficiently large that $\alpha^{\frac{2}{3}} \gg \varepsilon$. Within the region (3.2.6), the momentum fluxes are

$$\left.\begin{aligned} M^{(1)}(s) &= \alpha^4 \rho^{(1)} \int_{-\infty}^{0} (\overline{\partial \psi_{(2)}^{(v)}}^{(1)}/\partial n)^2 \, dn = \alpha^2 \varepsilon^{(1)} \rho^{(1)} \gamma^3 s, \\ M^{(2)}(s) &= \alpha^4 \rho^{(2)} \int_{0}^{\infty} (\overline{\partial \psi_{(2)}^{(v)}}^{(2)}/\partial n)^2 \, dn = \alpha^2 \varepsilon^{(2)} \rho^{(2)} \gamma^3 s, \end{aligned}\right\} \tag{3.2.8}$$

across any section s = constant of the outer boundary layers.

The mass transport velocity in the inner, oscillatory boundary layers is given by

$$\alpha^2 Q_s \simeq \alpha^{\frac{4}{3}} \gamma^2 s^{\frac{1}{3}} f'(0),$$

and is constant across these layers because the mean vorticity within the layers is $O(\alpha^2 \varepsilon^{-1})$.

4. The case of the air-water interface

In the important case when the upper and lower fluids are air and water, respectively, the solutions of Section 3 are not found to be adequate, except for very long waves. For the case of air above water,

$$\rho^{(2)}/\rho^{(1)} \ll 1, \quad \mu^{(2)}/\mu^{(1)} \ll 1, \quad \varepsilon^{(2)}/\varepsilon^{(1)} = O(1). \tag{4.1}$$

It has been shown by Dore [11] that the boundary condition (2.5) for the mean motion beyond the oscillatory layers must be replaced by

$$\left[\mu \frac{\partial^2 \Psi_l}{\partial n^2}\right] = 2^{\frac{1}{2}} \mu^{(2)}/\varepsilon^{(2)} + 4\mu^{(1)} \tag{4.2}$$

when the air-water interface is to be included in discussion. We now consider solutions of equation (3.1), taking account of the new boundary condition (4.2).

4.1. *The region near the origin*

A fundamental approximation is made in that the term $\mu^{(2)}(\partial^2 \Psi_l^{(2)}/\partial n^2)_{n_\infty^{(2)}}$ on the left-hand side of equation (4.2) is assumed to be much smaller than the remaining term. This assumption, which will be checked *a posteriori*, allows us to uncouple the boundary value problems in the air and water. Near $s = 0$, where the outer boundary layer is thin, the inviscid part of Q_s is approximately one. In the water, where (4.2) has become, by assumption, an *explicit* boundary condition for $\overline{\omega_{(2)}^{(1)}}$, considerations of order of magnitude show that the viscous part of Q_s must be negligible compared to unity, at least for sufficiently small s. Thus, by equation (3.1), the dominant part of $\overline{\omega_{(2)}^{(1)}}$ satisfies the diffusion equation

$$\frac{\partial \overline{\omega_{(2)}^{(1)}}}{\partial s} = \frac{\varepsilon^{(1)^2}}{\alpha^2} \frac{\partial^2 \overline{\omega_{(2)}^{(1)}}}{\partial n^2} \tag{4.1.1}$$

within the considered part of the layer. The boundary conditions on $\overline{\omega_{(2)}^{(1)}}$ are

$$\begin{aligned} &\overline{\omega_{(2)}^{(1)}} = 2^{\frac{1}{2}}\mu^{(2)}/\varepsilon^{(2)}\mu^{(1)} + 2, && n = 0, \quad s > 0, \\ &\overline{\omega_{(2)}^{(1)}} = 0 && s = 0, \quad n < 0, \\ &\overline{\omega_{(2)}^{(1)}} \to 0 && n \to -\infty, \; s > 0, \end{aligned} \tag{4.1.2}$$

and boundary condition (2.6) has been dropped. This omission is part of the fundamental approximation scheme referred to above, and must be made in order to obtain a well-determined boundary value problem for $\overline{\omega_{(2)}^{(1)}}$. The two terms on the right-hand side of the first condition here are equal when $\lambda = 84.7$ cm $= \lambda_0$. For longer (shorter) waves, the first (second) such term is dominant. We are thus able to obtain the similarity solution

$$\overline{\omega_{(2)}^{(1)}} = \left(\frac{2^{\frac{1}{2}}\mu^{(2)}}{\varepsilon^{(2)}\mu^{(1)}} + 2\right) \operatorname{erfc}\left(\frac{-\alpha n}{2\varepsilon^{(1)} s^{\frac{1}{2}}}\right). \tag{4.1.3}$$

The mean vorticity is confined within a second layer, of initially parabolic shape $\alpha^2 n^2/\varepsilon^{(1)^2} s = O(1)$, starting at the origin of the waves. The dominant viscous correction to the tangential component of mass transport velocity within the outer boundary layer may be obtained by integration of equation (4.1.3) with respect to n, together with use of the boundary condition $\overline{q_{s(2)}^{(v)(1)}} \to 0$ as $n \to -\infty$.

Considering the solution in the outer boundary layer (assumed thin) for the air, we must now satisfy the relevant form of condition (2.6). This suggests that the viscous part of $Q_s^{(2)}$ is $O(1)$ in the layer. Thus, (3.1) gives

$$\left(1 + \frac{\partial \overline{\psi_{(2)}^{(v)(2)}}}{\partial n}\right) \frac{\partial^3 \overline{\psi_{(2)}^{(v)(2)}}}{\partial s \partial n^2} - \frac{\partial \overline{\psi_{(2)}^{(v)(2)}}}{\partial s} \frac{\partial^3 \overline{\psi_{(2)}^{(v)(2)}}}{\partial n^3} = \frac{\varepsilon^{(2)^2}}{\alpha^2} \frac{\partial^4 \overline{\psi_{(2)}^{(v)(2)}}}{\partial n^4}, \tag{4.1.4}$$

subject to

$$\begin{aligned}
&\frac{\partial\overline{\psi^{(v)(2)}_{(2)}}}{\partial n} = 1, && n = 0, \quad s > 0, \\
&\overline{\psi^{(v)(2)}} = 0, && n = 0, \quad s > 0, \\
&\frac{\partial\overline{\psi^{(v)(2)}_{(2)}}}{\partial n} = 0, && s = 0, \quad n > 0, \\
&\frac{\partial\overline{\psi^{(v)(2)}_{(2)}}}{\partial n} \to 0, && n \to \infty \quad s > 0.
\end{aligned} \tag{4.1.5}$$

After an integration of equation (4.1.4) from n to ∞, we obtain the similarity solution

$$\overline{\psi^{(v)(2)}_{(2)}} + n = 2^{\frac{1}{2}}\varepsilon^{(2)}\alpha^{-1}s^{\frac{1}{2}}g(N^{(2)}), \quad (N^{(2)} = \alpha n/2^{\frac{1}{2}}\varepsilon^{(2)}s^{\frac{1}{2}}), \tag{4.1.6}$$

where $g(N^{(2)})$ satisfies the Blasius differential equation

$$g''' + gg'' = 0 \tag{4.1.7}$$

subject to the boundary conditions

$$g(0) = 0, \quad g'(0) = 2, \quad g'(\infty) = 1. \tag{4.1.8}$$

Coppel [12] has proved the existence of a solution satisfying this boundary value problem and the condition $g'' < 0$ for $0 \leq N^{(2)} < \infty$. Thus, in the air, the outer boundary layer has parabolic shape, starts at the origin of the waves, and has mass transport velocity

$$\alpha^2 Q_s^{(2)} \simeq \alpha^2 g',$$

which is everywhere in the direction of wave propagation. Beyond this layer, $Q_s^{(2)} = e^{-2n}$.

Breakdown of the solutions. Both solutions (4.1.3) and (4.1.6) break down close to the origin, notably, where $s = O(\varepsilon^2\alpha^{-2})$, s-derivatives are comparable with n-derivatives. When $\lambda \leq O(\lambda_0)$, both solutions are invalid at $s = O(\alpha^2\varepsilon^{-2})$, where the outer boundary-layer thicknesses have become $O(1)$, and the viscous part of the mean velocity, obtained from (4.1.3), is also $O(1)$. Similarly, when $\lambda \gg \lambda_0$, the solution (4.1.3) becomes invalid where $s = O(\alpha^2\mu^{(1)^2}/\mu^{(2)^2})$. Finally, to justify our approximate use of the boundary condition (4.2), we require

$$\begin{aligned}
&s \gg (\mu^{(2)}\alpha/\mu^{(1)}\varepsilon)^2 && \lambda \leq O(\lambda_0), \\
&s \gg \alpha^2 && \lambda \gg \lambda_0.
\end{aligned} \tag{4.1.9}$$

Thus, for $\lambda \leq O(\lambda_0)$, there is always a range, such that

$$\max(\varepsilon^2\alpha^{-2}, \mu^{(2)^2}\alpha^2/\mu^{(1)^2}\varepsilon^2) \ll s \ll \alpha^2\varepsilon^{-2},$$

in which the solutions (4.1.3) and (4.1.6) are valid approximations. (It is implicit here that we are assuming that the wave amplitude is sufficiently large that $\alpha \gg \varepsilon$.) If $\lambda \gg \lambda_0$, there is, in general, a range, such that

$$\max(\varepsilon^2\alpha^{-2}, \alpha^2) \ll s \ll \mu^{(1)^2}\alpha^2/\mu^{(2)^2},$$

in which these solutions are useful. Of course, when $\lambda \ll \lambda_0$, the solution (4.1.3) in the water agrees closely with that obtained in I, wherein the presence of the air is neglected.

4.2. *The intermediate and far fields*

As explained above, the approximate solutions of Section 4.1 become invalid where

$$\begin{aligned} s &= O(\alpha^2\varepsilon^{-2}) \gg 1 & \lambda &\leq O(\lambda_0), \\ s &= O(\mu^{(1)^2}\alpha^2/\mu^{(2)^2}) \ll O(\alpha^2\varepsilon^{-2}) & \lambda &\gg \lambda_0. \end{aligned} \tag{4.2.1}$$

When $\lambda \leqq O(\lambda_0)$, the situation is analogous to that described in I. The region (4.2.1) of invalidity of the solutions of Section 4.1 has outer boundary-layer thicknesses $O(1)$. This region will be designated the intermediate field, and has $\partial\overline{\psi_{(2)}^{(v)}}/\partial n = O(1)$ in both air and water. Within the region, it is necessary to study the full, non-linear version of equation (3.1), in which the inviscid part of Q_s is e^{-2n} (e^{2n}) in the air (water). The boundary value problems in the air and water are coupled by the boundary conditions arising from (2.5) and (2.6), but can be uncoupled by the assumption made about (4.2) at the beginning of Section 4.1. However, analytical solution presents considerable difficulties, and no numerical solution is worthwhile here. As in I, we make use of the fact that the thicknesses of the outer boundary layers continue to increase with s. Hence $\partial\overline{\psi_{(2)}^{(v)(1)}}/\partial n$ increases with s, eventually becoming $\gg 1$; by condition (2.6), $\partial\overline{\psi_{(2)}^{(v)(2)}}/\partial n$ must also become $\gg 1$. The region where this holds is termed the far field. On neglecting the inviscid part of Q_s in equations (3.1) in the far field, we have an equation of the form (3.2.1) in each fluid. The boundary conditions are the same as those of (3.2.2), except that the first condition is replaced by

$$\mu^{(1)}\frac{\partial^2\overline{\psi_{(2)}^{(v)(1)}}}{\partial n^2} - \mu^{(2)}\frac{\partial^2\overline{\psi_{(2)}^{(v)(2)}}}{\partial n^2} = 2^{\frac{1}{2}}\mu^{(2)}/\varepsilon^{(2)} + 2\mu^{(1)}, \qquad n = 0. \tag{4.2.2}$$

As in the corresponding stage of Section 3.2, we are able to obtain the similarity solution

$$\begin{aligned} \overline{\psi_{(2)}^{(v)(1)}} &= \frac{\varepsilon^{(1)}}{\alpha^{\frac{2}{3}}}\beta s^{\frac{2}{3}} f\left(\frac{\alpha^{\frac{2}{3}}}{\varepsilon^{(1)}}\frac{\beta n}{s^{\frac{1}{3}}}\right), \\ \overline{\psi_{(2)}^{(v)(2)}} &= -\frac{\varepsilon^{(2)}}{\alpha^{\frac{2}{3}}}\beta s^{\frac{2}{3}} f\left(-\frac{\alpha^{\frac{2}{3}}}{\varepsilon^{(2)}}\frac{\beta n}{s^{\frac{1}{3}}}\right), \end{aligned} \tag{4.2.3}$$

where

$$\beta = \left\{\frac{2^{\frac{1}{2}}(\rho^{(2)}\mu^{(2)})^{\frac{1}{2}} + 2\varepsilon^{(1)}(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}}\right\}^{\frac{1}{3}},$$

and $f(N)$ is the solution of boundary value problem (3.2.4). The region of the far field is therefore $s \gg \alpha^2 \varepsilon^{-2}$. In view of (4.1), it is seen that the first term on the left-hand side of (4.2.2) dominates the second term, which could therefore have been neglected. This is equivalent to neglecting the term $(\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}$ in the denominator of β. If this is done, and if $\lambda \ll \lambda_0$, we have $\beta \approx (2\varepsilon^{(1)})^{\frac{1}{3}}$, and then the corresponding expression for $\overline{\psi_{(2)}^{(v)\,(1)}}$ is in agreement with equations (2.2.3), (2.2.4) of I.

When $\lambda \gg \lambda_0$, it is found that there is a region $s = O(\alpha^2 \mu^{(1)^2}/\mu^{(2)^2})$, termed the intermediate field, in which the thickness $O(\varepsilon\mu^{(1)}/\mu^{(2)})$ of the outer boundary layers is still small, and the solution of Section 4.1 is invalid. In such a region, $\partial\overline{\psi_{(2)}^{(v)}}/\partial n$ is $O(1)$ in both fluids. Thus $\overline{\psi_{(2)}^{(v)\,(1)}}$ satisfies an equation of the same form as (4.1.4), but with $\varepsilon^{(2)}$ replaced by $\varepsilon^{(1)}$. The appropriate boundary conditions are

$$\begin{aligned}
&\frac{\partial^2 \overline{\psi_{(2)}^{(v)\,(1)}}}{\partial n^2} = 2^{\frac{1}{2}}\mu^{(2)}/\varepsilon^{(2)}\mu^{(1)} + 2 \approx 2^{\frac{1}{2}}\mu^{(2)}/\varepsilon^{(2)}\mu^{(1)}, && n = 0,\\
&\overline{\psi_{(2)}^{(v)\,(1)}} = 0, && n = 0, \\
&\frac{\partial \overline{\psi_{(2)}^{(v)\,(1)}}}{\partial n} \to 0, && n \to -\infty,
\end{aligned} \tag{4.2.4}$$

together with an assumed known distribution of $\overline{\psi_{(2)}^{(v)\,(1)}}$ at a section $s =$ constant at the left of the region. The non-homogeneous condition in (4.2.4) is representative of a mean tangential stress, from left to right, acting on the water in the outer boundary layer. Within the double layer in the air, $\overline{\psi_{(2)}^{(v)\,(2)}}$ satisfies equation (4.1.4), together with the boundary conditions

$$\begin{aligned}
&\frac{\partial \overline{\psi_{(2)}^{(v)\,(2)}}}{\partial n} = 1 + \frac{\partial \overline{\psi_{(2)}^{(v)\,(1)}}}{\partial n}, && n = 0,\\
&\overline{\psi_{(2)}^{(v)\,(2)}} = 0, && n = 0, \\
&\frac{\partial \overline{\psi_{(2)}^{(v)\,(2)}}}{\partial n} \to 0, && n \to \infty,
\end{aligned} \tag{4.2.5}$$

and a starting condition at some appropriate section $s =$ constant. Thus, in principle, it is possible to solve the boundary value problems for $\overline{\psi_{(2)}^{(v)\,(1)}}$, $\overline{\psi_{(2)}^{(v)\,(2)}}$ consecutively, beginning with that for $\overline{\psi_{(2)}^{(v)\,(1)}}$. The condition that the boundary layer assumptions should hold is that $\mu^{(2)}\varepsilon/\mu^{(1)} \ll \alpha^2$. Where $s \gg \alpha^2\mu^{(1)^2}/\mu^{(2)^2}$, we have the region of the far field. Here, the solutions for $\overline{\psi_{(2)}^{(v)\,(1)}}$, $\overline{\psi_{(2)}^{(v)\,(2)}}$ are similar to the far-field solutions in the case $\lambda \leq O(\lambda_0)$. That is, equation (4.2.3) applies, but β may be written approximately as

$$\beta \approx \{2^{\frac{1}{2}}(\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}/(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}}\}^{\frac{1}{3}}. \tag{4.2.6}$$

This means that the solution is given by the solution of the problem of Section 3, provided that, in the latter solution, the approximations of (4.1) are made.

Breakdown of the far-field solutions. If $\lambda \leq O(\lambda_0)$, $\partial\overline{\psi^{(v)}_{(2)}}/\partial n = O(\varepsilon^{\frac{2}{3}}s^{\frac{1}{3}}/\alpha^{\frac{2}{3}})$, so that the theory breaks down where $s = O(\alpha^{-1}\varepsilon^{-2})$, in which region mean and primary oscillatory velocities would be comparable. This is similar to the corresponding work of I. If $\lambda \gg \lambda_0$, $\partial\overline{\psi^{(v)}_{(2)}}/\partial n = O(\mu^{(2)\frac{2}{3}}s^{\frac{1}{3}}/\mu^{(1)\frac{2}{3}}\alpha^{\frac{2}{3}})$ and breakdown occurs where $s = O(\mu^{(1)^2}/\mu^{(2)^2}\alpha)$. The solutions in the far field are therefore valid in the regions

$$\begin{aligned} &\alpha^2\varepsilon^{-2} \ll s \ll \alpha^{-1}\varepsilon^{-2} \qquad && \lambda \leq O(\lambda_0), \\ &\alpha^2\mu^{(1)^2}/\mu^{(2)^2} \ll s \ll \mu^{(1)^2}/\mu^{(2)^2}\alpha \qquad && \lambda \gg \lambda_0. \end{aligned} \tag{4.2.7}$$

With $\alpha \gg \varepsilon$, validity of the boundary-layer assumptions are guaranteed throughout the region when $\lambda \leqq O(\lambda_0)$, and there is an analogous result when $\lambda \gg \lambda_0$ provided that $\alpha^2\mu^{(1)} \gg \varepsilon\mu^{(2)}$. (If $\varepsilon \gg \alpha$, which is less likely in practice, the far-field solutions are still valid sufficiently far from the generating region.)

5. Concluding remarks

Using the notion of outer boundary layers, an approximate mathematical model has been given of the mass transport velocity in such layers when a progressive wave propagates on the interface between two semi-infinite fluids.

1) When the densities and viscosities are comparable, very large mass transport velocities are achieved over relatively short distances, but the theory becomes invalid where $s = O(\alpha^{-1})$.

2) In the case of the air-water interface, there are essentially three régimes of wavelengths to consider.

(i) When $\lambda \ll \lambda_0$, the solution in the water is basically the same as that given in I, where the presence of the air is neglected. The theory becomes invalid where $s = O(\alpha^{-1}\varepsilon^{-2})$.

(ii) When $\lambda = O(\lambda_0)$, the solution in the water is an elementary modification of that of (i), but the effects of the air are now significant.

(iii) When $\lambda \gg \lambda_0$, the effects of the air are *dominant* and the far-field solutions may be derived from the far-field solutions of 1), provided that, in the latter, the approximations $\rho^{(2)} \ll \rho^{(1)}$, $\mu^{(2)} \ll \mu^{(1)}$ are made. The validity of the theory, however, breaks down farther from the origin, at $s = O(\mu^{(1)^2}/\mu^{(2)^2}\alpha)$.

In all cases, the tangential component of mass transport velocity at a fixed level increases continuously with distance from the generating region. Additionally, the mass transport velocity in the far field dominates the Stokes drift velocity $O(\alpha^2)$.

Note added in processing. In recent work, Liu and Davis [13] have, *inter alia*, considered the mass transport velocity due to progressive waves in a homogeneous, viscous fluid of uniform depth h. Effects of the air are ignored. The waves are assumed to be decaying with time over the time-scale $O(\varepsilon^{-2})$. However, the correct time-scale is $O(\varepsilon^{-1})$ when $kh = O(1)$. Thus, the proposed solution is, unfortunately, erroneous, since the time-scale required for secondary, mean vorticity to diffuse (as they assume) over the whole depth is much greater, viz., $O(\varepsilon^{-2})$.

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